Exam Lie Groups in Physics

Date	November 4, 2014
Room	V5161.0293
Time	9:00 - 12:00
Lecturer	D. Boer

- Write your name and student number on every separate sheet of paper
- Raise your hand for more paper
- You are not allowed to use the lecture notes, nor other notes or books
- The weights of the three problems are given below
- Answers may be given in Dutch
- Illegible handwriting will be graded as incorrect
- Good luck!

Weighting

1a)	9	2a)	9	3a)	9
1b)	9	2b)	9	3b)	9
1c)	9	2c)	5	3c)	9
1d)	4	2d)	5	3d)	4

$$\text{Result} = \frac{\sum \text{points}}{10} + 1$$

Problem 1

Consider the Lie group $SU(3)$ of unitary 3×3 matrices with determinant 1.

(a) Derive the dimension of $SU(3)$.

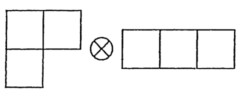
(b) Indicate a way in which $SU(2)$ can be viewed as a subgroup of $SU(3)$ and explain why $SU(3)/SU(2)$ does not form a group.

(c) Determine the center of $SU(3)$ (you may assume that the defining rep is an irrep).

(d) Give an example from physics where $SU(3)$ plays a role.

Problem 2

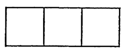
(a) Decompose the following direct product of irreps of the Lie algebra $su(n)$



into a direct sum of irreps of $su(n)$.

(b) Write down the dimensions of the irreps appearing in the obtained decomposition for $su(2)$ and $su(3)$.

(c) Relate the decomposition for $su(2)$ to the corresponding case of addition of angular momentum in Quantum Mechanics.

(d) Write down the complex conjugate diagram of  for $su(3)$ and verify its dimension.

Problem 3

Consider the Lie group $E^+(2)$ of proper Euclidean transformations in two dimensions that consists of elements $(R|\vec{a})$, where $R \in SO(2)$ and $\vec{a} = (a_x, a_y) \in \mathsf{R}^2$. An element $(R|\vec{a})$ acts on a two-vector $\vec{r}$ as a rotation followed by a translation: $(R|\vec{a})\vec{r} = R\vec{r} + \vec{a}$.

(a) Derive the composition law in $E^+(2)$, in other words, specify R_3 and $\vec{a}_3$ in $(R_1|\vec{a}_1)(R_2|\vec{a}_2) = (R_3|\vec{a}_3)$. Furthermore, show that this multiplication rule is satisfied by the matrix representation:

$$D\left((R|\vec{a})\right) = \begin{pmatrix} R_{11} & R_{12} & a_x \\ R_{21} & R_{22} & a_y \\ 0 & 0 & 1 \end{pmatrix},$$

and show that this is a reducible representation by indicating an invariant subspace of the carrier space.

(b) Consider in the above three-dimensional representation D those elements of $E^+(2)$ that are infinitesimally close to the 3×3 identity matrix and use them to derive the Lie algebra of $E^+(2)$:

$$\begin{aligned} [\lambda, N_1] &= iN_2, \\ [\lambda, N_2] &= -iN_1, \\ [N_1, N_2] &= 0. \end{aligned}$$

(c) Derive the quadratic Casimir operator of $E^+(2)$.

(d) Explain the role of $E^+(2)$ in the analysis of the representations of the Poincaré group.